International Journal of Mathematics and Computer Applications Research (IJMCAR)
ISSN(P): 2249-6955; ISSN(E): 2249-8060
Vol. 4, Issue 1, Feb 2014, 9-18

SET PACKING PROBLEM WITH LINEAR FRACTIONAL OBJECTIVE FUNCTION

RASHMI GUPTA[1] & RATNESH RAJAN SAXENA[2]

[1]Research Scholar, Department of Mathematics, University of Delhi, India

[2]Associate Professor, Department of Mathematics, Deen Dayal Upadhyaya College, University of Delhi, India

ABSTRACT

The Set Packing problem has a dual covering problem, which asks how many of the same objects are required to completely cover every region of the container, where the objects are allowed to overlap. Many applications arise having the packing and covering structure. Delivery and routing problems, scheduling problems and location problems, switching theory, wireless network design, VLSI circuits and line balancing often take on a set covering structure. However, if one wishes to satisfy as much demand as possible without creating conflict, it takes on a set packing format. In this paper a linearization technique is developed to solve set packing problems with linear fractional objective function. The correctness of the algorithm is shown by an numerical example.

AMS Subject Classifications: 90C10, 90C20.

KEYWORDS: Set Packing Problem, Linear Fractional Set Packing Problem

1. INTRODUCTION

The Set Packing problems are the interesting topic for many of the researchers but, surprisingly, few applications of the Set Packing formulation have been reported in the literature. Some of them are described below. Ronnqvist [19] worked on a cutting stock problem formulated as Set Packing Problem and solved using a Lagrangian relaxation combined with subgradeint optimization. Zwaneveld et al. [24] formulated a real railway feasibility problem as Set Packing and solved it exactly using reduction tests and a Branch & Cut method. Kim [12] represented a ship scheduling problem as Set Packing and used LINDO software to solve it. Mingozzi et al. [14] used an SPP formulation to calculate the bounds for a Resource Constrained Project scheduling Problem using a greedy method. Rossi [20] considered an SPP formulation for a ground holding problem and solved it exactly with a Branch & Cut method.

The railway industry is rich in problem that can be modelled and solved using Set Packing format. In this day and age, arguably the most important of these are the ones that concern the effective allocation and utilization of available resources. The problem of routing trains through railway junctions arises at each of the level. Railway management often face the task of deciding between a number of possible investment alternatives concerning proposed infrastructure modifications to junctions of which the most influential factor in making the final decision is capacity. Railway management are very interested in knowing, with precision, what level of rail traffic the modified infrastructure would cater for. This effectively involves determining the maximum number of trains that could be routed through the junction within a given time horizon. Scheduling airline flight crews to airplanes is another application of set packing. Each airplane in the fleet needs to have a crew assigned to it, consisting of a pilot, copilot, and navigator. There are constraints on the composition of possible crews, based on their training to fly different types of aircraft, as well as any personality conflicts. Given all possible crew and plane combinations, each represented by a subset of items, we need an assignment such that each plane and each person is in exactly one chosen combination. After all, the same person cannot be on two different planes, and every plane needs a crew. We need a perfect packing given the subset constraints.

There are many applications of fractional packing problems in real life as well. Some of them are 'air line crew scheduling', 'truck routing', 'political districting', 'information retrieval', etc. For example, suppose an air line company has 'm' flights, $I = \{1, 2, \ldots, m\}$ to operate upon and 'n' crews, $J = \{1, 2, \ldots, n\}$ at its disposal, it being understood that a crew can handle at least one flight. Let $c_j > 0$ be the profit earned by the company when its j^{th} crew is in operation and let d_j be the some utility of the j^{th} crew paid by the company. Also $\alpha > 0$ be the fixed value of this utility function. Now the company is interested in scheduling its crew in such a way that all the flights are covered and the ratio of the earning and the utility function is maximized.

If a variable x_j is defined as

$x_j = \begin{cases} 1 \; if \; j^{th} \; crew \; is \; in \; schedule \\ 0 \;\; otherwise \end{cases}$ and $a_{ij} = \begin{cases} 1 & if \; i^{th} \; flight \, by \; j^{th} \; crew \\ 0 & otherwise \end{cases}$, then fractional set packing problem is to find a set of crews $J^* \subseteq J$ that covers all the flights and $Maximize \dfrac{\sum c_j x_j}{\sum d_j x_j + \alpha}$; $j \in J$.

2. THEORETICAL DEVELOPMENT

2.1 Set Packing Problems

Consider a set $I = \{1, 2, ..., m\}$ and set $P = \{P_1, P_2, ..., P_n\}$ where $P_j \subseteq I$ and $j \in J = \{1, 2, ..., n\}$. A subset J^* of J is said to be a pack of I if $\bigcup_{j \in J^*} P_j = I$, and $j \,\&\, k \in J^*$, $j \neq k$ and $P_j \bigcap P_k = \phi$. Let a weight $c_j > 0$ be associated with every $j \in J$. The total weight of the packing J^* is equal to $\sum_{j \in J^*} c_j$.

The linear set packing problem **(PkP)** is to find a pack of maximum weight subject to the condition that at the most one of the utility is satisfied. Mathematically the problem is

(PkP) $Max \;\; f(x) = \sum_{j=1}^{n} c_j x_j$

Subject to $\qquad \sum_{j=1}^{n} a_{ij} x_j \leq 1, \; i \in I$ (2.1)

$x_j = 0 \; or \; 1, \;\; j \in J$ (2.2)

Where $x_j = \begin{cases} 1 & \text{if j is in the pack} \\ 0 & \text{otherwise} \end{cases}$ and $a_{ij} = \begin{cases} 1 & if \; i \in P_j \\ 0 & otherwise \end{cases}$

Analytically, the Linear Fractional Set Packing Problem **(LFP)** with the same restrictions is to find a pack of maximum weight: therefore, mathematically, the problem is-

(LFP) $Max\ f(x) = \dfrac{\sum_{j=1}^{n} c_j x_j}{\sum_{j=1}^{n} d_j x_j + \alpha}$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \le 1,\ i \in I$ (2.3)

$x_j = 0\ or\ 1,\ j \in J$ (2.4)

Where $x_j = \begin{cases} 1 & \text{if j is in the packing} \\ 0 & \text{otherwise} \end{cases}$ and $a_{ij} = \begin{cases} 1 & if\ i \in P_j \\ 0 & otherwise \end{cases}$

It is assumed that c_j 's and d_j 's are non-negative and α is a scalar such that $\sum_{j=1}^{n} d_j x_j + \alpha > 0$. In matrix form, **(LFP)** can be written as

$$\begin{aligned} &Maximize\ f(x) = \frac{Cx}{Dx + \alpha}\ . \\ &subject\ to\ Ax \le b \end{aligned}$$

Where $x^T = (x_1, x_2, ..., x_n)\ with\ x_j = 0\ or\ 1,\ j = 1, 2, ..., n.$ Here $C = (c_1, c_2, ..., c_n) \in R^n$ and $D = (d_1, d_2, ..., d_n) \in R^n$ are row vectors, A is an m×n matrix of zeros and ones and $b^T = (1,1,...,1)$ is a row vector of ones.

DEFINITIONS

- **Pack Solution:** A solution x which satisfies **(2.1)** and **(2.2)** is said to be a pack solution.
- **Redundant Pack:** For any pack $J,$ a column $j^* \in J$ is said to be redundant if $J + \{j^*\}$ is also a pack.
- If a pack contains one or more redundant columns, it is called a redundant pack.
- **Prime Pack Solution:** A pack J^* is said to be a prime pack, if none of the columns corresponding to $j \in J^*$ is redundant. A solution corresponding to the prime pack is called a prime pack solution.
- **Pseudo concave Function:** Let f be a differentiable function defined on an open set $T \subset R^n$. Let $S \subset T$ and $X_1, X_2 \in S,$ then f is said to be pseudo concave if $\nabla f(X_2)^T (X_1 - X_2) \le 0 \Rightarrow f(X_1) \le f(X_2).$

2.1.2 Following are the Results that Form the Basis of the Algorithm to Enumerate the Problem

Theorem 1.[5] *If* $J^* = \{j : x_j = 1\}$ *is any prime pack of* ***(LFP)*** *then* $x = \{x_j\}$ *is an extreme point of the convex set formed by feasible region.*

Theorem 2. *If the objective function in* ***(LFP)*** *has finite value then, there exists a prime pack solution where this value is attained.*

Proof: Let a finite optimal solution of (**LFP**) exists at $x_0 \in S$ then the optimal value is

$$f(x_0) = \frac{Cx_0}{Dx_0 + \alpha}.$$

Let J_0 be the pack corresponding to the solution x_0.If J_0 is the prime pack, then it is done, otherwise a prime pack can be derived from J_0 by considering the redundant columns. Let J_1 be the prime pack obtained from J_0 and x_1 be the corresponding solution of **(LFP)** such that

$$f(x_1) = \frac{Cx_1}{Dx_1 + \alpha}.$$

Since $C, D \geq 0$, α is the positive and $J_1 \supseteq J_0$, therefore,

$$\frac{Cx_1}{Dx_1 + \alpha} \geq \frac{Cx_0}{Dx_0 + \alpha}$$

Or $f(x_1) \geq f(x_0)$

As $f(x_0)$ is the optimal value of $f(x)$, therefore, $f(x_1) \leq f(x_0)$. Hence $f(x_1) = f(x_0)$. Which proves that there exists a prime pack solution yielding the optimal value of the objective function of **(LFP)**.

<u>Theorem 3.</u> *Let $f(x)$ be a pseudo concave function defined on feasible set S and $x^* \in S$ then x^* is an optimal solution for the program*

$$\underset{x \in S}{Maximize}\ f(x)$$

if and only if, x^* is an optimal solution for the program

$$\underset{x \in S}{Maximize}\ \nabla f(x^*)^T x$$

where S is the feasible region.

Proof: Let x^* be an optimal solution for the program **(LFP)**, therefore, $f(x^*) \geq f(x)$, $\forall\, x \in S$.

As f is differentiable at x^*, therefore,

$$f(x) = f(x^* + x - x^*)$$

$$= f(x^*) + \nabla f(x^*)^T (x - x^*) + \alpha(x^*, x - x^*)\left|x - x^*\right|$$

Where $\alpha(x^*, x - x^*) \to 0$ as $x \to x^*$

Since $f(x^*) \geq f(x) \Rightarrow \nabla f(x^*)^T (x - x^*) + \alpha(x^*, x - x^*)\left|x - x^*\right| \leq 0$

And $\alpha(x^*, x - x^*) \to 0$

$$\Rightarrow \nabla f(x^*)^T (x - x^*) \le 0$$

$$\Rightarrow \nabla f(x^*)^T x \le \nabla f(x^*)^T x^*$$

$\Rightarrow x^*$ is an optimal solution for the program $Max \nabla f(x^*)^T x$.

Conversely, let x^* be an optimal solution of $Max \nabla f(x^*)^T x$ therefore, $\nabla f(x^*)^T x^* \ge \nabla f(x^*)^T x, \forall x \in S$

$$\Rightarrow \nabla f(x^*)(x - x^*) \le 0, \forall x \in S$$

Since f is a pseudo concave function, therefore, $f(x) \le f(x^*), \forall x \in S.$

Hence x^* is an optimal solution for **(LFP)**.

Following is the algorithm developed to enumerate the given set packing problem.

ALGORITHM

Step 1: Consider a Linear Fractional Set Packing Problem **(LFP)**. Form the corresponding continuous program **(LFP')** by embedding the feasible region into R^n (a cube with n vertices). Let *S* be the feasible set for **(LFP').**

Step 2: Choose a feasible solution $x_0 \in S$ such that $\nabla f(x_0) \ne 0.$ Form the corresponding linear program **(LP)**.On solving **(LP)** let x_1 be its optimal solution. If $x_1 = x_0$ then this is the required solution of the given problem, otherwise let $S_1 = \{x_1\}$.

Step 3: Starting with the point x_1, form corresponding **(LP)**, let its optimal solution be $x_2 \ne x_1$. Update S_1 i.e. $S_1 = \{x_1, x_2\}$.

Step 4: Repeat step 3 for the point x_2 and suppose at the ith stage $S_i = \{x_1, x_2, ..., x_i\}$. Stop if at the (i+1)th stage $x_{i+1} \in S_i$, then x_{i+1} is the optimal solution of **(LFP).**

Step 5: If x_{i+1} is an optimal solution of the form 0-1 then it is a solution of **(LFP)** otherwise, go to **Step-6**.

Step 6: Apply Gomory cuts to find a solution of the 0-1 form and the corresponding prime cover.

Note: The algorithm must terminate after finite number of steps as it moves only on the vertices of the feasible cube R^n, which are finite in numbers, i.e. convergence is must.

NUMERICAL EXAMPLE

$$\text{(LFP)}\quad \begin{aligned} &Max\ \ f(x) = \frac{2x_1 + 9x_2 + 6x_3}{x_1 + 4x_2 + x_3 + 2} \\ &subject\ to \quad x_1 + x_2 \le 1 \\ &\qquad x_2 + x_3 \le 1 \\ &\qquad x_1 + x_3 \le 1 \\ &\qquad x_1, x_2, x_3 = 0\ or\ 1 \end{aligned}$$

where $J = \{1,2,3\}$, $I = \{1,2,3\}$

Step 1: The corresponding **(LFP')** is

$$Max\ \ f(x) = \frac{2x_1 + 9x_2 + 6x_3}{x_1 + 4x_2 + x_3 + 2}$$

$$x = (x_1, x_2, x_3) \in S = \{(x_1, x_2, x_3) \mid x_1 + x_2 \le 1, x_2 + x_3 \le 1, x_1 + x_3 \le 1,\ x_1, x_2, x_3 \ge 0\}$$

Step 2: Choose $x_0 = (1,0,0)$ as one of the feasible solution of **(LFP')** with $\nabla f(x_0) \ne 0$

The corresponding **(LP)** is

$$Maximize\ \nabla f(x_0)^T x = \frac{4}{9}x_1 + \frac{19}{9}x_2 + \frac{16}{9}x_3 : x \in S$$

Now it is a linear problem therefore can be solved by simplex method i.e.

After applying the simplex algorithm the final optimal table is as follows:

Table 1

		C_j	4/9	19/9	16/9	0	0	0
C_B	B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3
19/9	X_2	½	0	1	0	½	½	-1/2
16/9	X_3	½	0	0	1	-1/2	½	½
4/9	X_1	½	1	0	0	½	-1/2	½
		39/18	0	0	0	7/18	31/18	1/18

Hence the optimal solution is $(1/2, 1/2, 1/2)$, which is not of the form 0 or 1, therefore apply the Gomory cut to get integer solution.

The cut is $S_4 = \frac{-1}{2} + \frac{1}{2}x_4 + \frac{1}{2}x_5 + \frac{1}{2}x_6$

Now inserting this additional constraint in the optimal simplex table the next iterative table is

Table 2

		C_j	4/9	19/9	16/9	0	0	0	0
C_B	B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	S_4
19/9	X_2	½	0	1	0	1/2	½	-1/2	0
16/9	X_3	½	0	0	1	-1/2	½	½	0
4/9	X_1	½	1	0	0	1/2	-1/2	½	0
0	S_4	-1/2	0	0	0	-1/2	-1/2	-1/2	1
		39/18	0	0	0	7/18	31/18	1/18	0

Now apply the dual simplex method, drop S_4 and enter S_3 next iterative table is

Table 3

			4/9	19/9	16/9	0	0	0	0
C_B	B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	S_4
19/9	Y_2	1	0	1	0	1	1	0	-1
16/9	Y_3	0	0	0	1	-1	0	0	1
4/9	Y_1	0	1	0	0	0	-1	0	1
0	S_3	1	0	0	0	1	1	1	-2
		19/9	0	0	0	1/3	5/3	0	1/9

Hence the optimal solution of **(LP)** is $x_1 = (0,1,0)$, which is not equal to $x_0 = (1,0,0)$ therefore let $S_1 = \{ x_1 \}$.

Step 3: Now starting with the point X_1 the corresponding **(LP)** is

$$Maximize\ \nabla f(x_1)^T x = \frac{3}{36}x_1 + \frac{18}{36}x_2 + \frac{27}{36}x_3 : x \in S$$

Now it is a linear problem therefore can be solved by simplex method.

After applying the simplex algorithm the final optimal table is as follow:

Table 4

		C_j	3/36	18/36	27/36	0	0	0
C_B	B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3
0	S_1	1	0	2	0	1	1	-1
27/36	X_3	1	0	1	1	0	1	0
3/36	X_1	0	1	-1	0	0	-1	1
		27/36	0	6/36	0	0	24/36	3/36

Hence the optimal solution of **(LP)** is $x_2 = (0,0,1)$, which is not equal to $x_1 = (0,1,0)$, therefore, let $S_2 = \{ x_1, x_2 \}$.

Now starting with the point x_2 the corresponding **(LP)** is

$$Maximize\ \nabla f(x_2)^T x = \frac{3}{9}x_2 + \frac{12}{9}x_3 : x \in S$$

Now it is a linear problem therefore can be solved by simplex method.

After applying the simplex algorithm the final optimal table is as follows:

Table 5

		C_j	0	3/9	12/9	0	0	0
C_B	B	X_B	Y_1	Y_2	Y_3	S_1	S_2	S_3
0	S_1	1	1	1	0	1	0	0
12/9	X_3	1	0	1	1	0	1	0
0	S_3	0	1	-1	0	0	-1	1
		12/9	0	1	0	0	12/9	0

Hence the optimal solution of **(LP)** is $x_3 = (0,0,1)$, which is equal to $x_2 = (0,0,1)$ therefore this is the optimal solution for the original **(LFP)** with optimal value 2.

CONCLUDING REMARKS

In this paper we have considered the Set Packing Problem with linear fractional objective function the technique developed is enumerative one which go through the vertices of the given problem and gives the optimal solution. Most importantly the algorithm just needs the pseudo-concavity of the objective function.

REFERENCES

1. Arora, S.R. and Puri, M.C. (1977): “Enumeration Technique for the Set Covering Problem with Linear Fractional Functional as its Objective Functions”, ZAMM, 56, 181-186.
2. Bellmore, M. and Retliff, H. D. (1971): “Set Covering and Involutary Basis”, Management Sciences, 18, p.194-206.
3. Balas, E. (1980): “Cutting Planes from Conditional Bounds, A New Approach Set Covering”, Mathematical Programming Study, 12, 19-36.
4. Balas, E. and Padberg, M. W. (1976): “Set Partitioning: A Survey”, SIAM Review, 18, 710-760.
5. Bazaraa et. al. (1993): “Non Linear Programming Theory and algorithms”, 2nd edition John Wiley & Sons.
6. Chavatal, V. (1979): “A Greedy Heuristic for the Set Covering Problem", Mathematics of Operations Research, 4, 233-235.
7. Fisher, M. and Wolsey, L. (1982): “On the Greedy Heuristic for Covering and Packing Problems”, SIAM Journal on Algebraic and Discrete Methods, 3, 584-591.
8. Feo, A. and Mauricis and Resende, G. C. (1989): “A Probabilistic Heuristic for a Computationally Difficult Set Covering Problem”, Operations Research Letters, 8, 67-71.
9. Garfinkel, R.S. and Nemhauser, G.L. (1973): “Integer Programming”, A Wiley Inter Science Publication, John Wiley and Sons.
10. Hall, N. and Hochbaum, D. (1989): “A Fast Approximation Algorithm for the Multi Covering Problem”, Discrete Applied Mathematics, 15, 35-40.
11. Huang, W.C., Kao, C.Y. and Hong, J.T. (1994) “A Genetic Algorithm for Set Covering Problems”, IEEE International Conference on Genetic Algorithms: Proceedings, 569-574.
12. Kim, S.-H. and Lee, K.-K. (1997): “An Optimization Based Decision Support System for Ship Scheduling”, Computers and Industrial Engineering 33, 689-692.
13. Lemke, C.E., Salkin, H.M. and Spielberg, K. (1971): “Set Covering by Single Branch Enumeration with Linear Programming Sub Problem”, Operations Research, 19, 998-1022.
14. Mingozzi, A., Maniezzo, V., Ricciardelli, S. and Bianco, L. (1998): “An Exact Algorithm for the Project Scheduling with Resource Constraints Based on a New Mathematical Formulation”, Management Science. 44, 714-729.
15. Nemhauser, G.L, Wolsey, L.A (1999): “Integer and Combinatorial Optimization”, Wiley Interscience Series in Discrete Mathematics and Optimization.
16. Padberg, M.W (1993): “Lehman’s Forbidden Minor Characterization of Ideal 0-1 Matrices”, Discrete Mathematics, 111, 409-410.

17. Padberg, M.W. (1974):"Perfect Zero-One Matrices", Mathematical Programming, 6, 180-196.
18. Roth, R. (1969): "Computer Solution to Minimum Cover Problem", Operations Research, 17, 455-465
19. Ronnqvist, M. (1995): "A Method for the Cutting Stock Problem with Different Qualities", European Journal of Operational Research 83, 57-68.
20. Rossi, F. and Smriglio, S. (2001): "A Set Packing Model for the Ground Holding Problem in Conjested Networks", European Journal of Operational Research, 131, 400-416.
21. Saxena, R.R. and Arora, S.R. (1998): "Cutting Plane Technique for the Multi-Objective Set Covering Problem with Linear Fractional Objective Functions", IJOMAS, 14, 1, 111-122.
22. Saxena, R.R. and Arora, S.R. (1996): "A Linearization Technique for Solving the Quadratic Set Covering Problem", Optimization, 39, 1, 35-42.
23. Saxena, R.R. and Arora, S.R. (1995): "Enumeration Technique for Solving Multi Objective Linear Set Covering Problem", APZOR, 12, 87-97.
24. Zwaneveld, P.J., Kroon, L.G., Romeijn, H.E., Salomon, M., Dauzere-Peres, S., Van Hoesel, S.P. and Ambergen, H.W. (1996): "Routing Trains Through Railway Stations: Model Formulation and Algorithms", Transportation Science 30, 3, 181-194.